Product of distributions in $BV(\mathbb{R})$.

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Abstract

In this paper it will be shown how the description of the *electrical regime* of a simple electric circuit leads to precise definition of some product of distributions in $BV(\mathbb{R})$, the space of functions of bounded variation whose domain is $\mathbb{R}$. Precisely, it became possible to define the product $H(t) \cdot \delta(t)$ as a *Radon measure*.

1 Description of the electric circuit under analysis and statement of the related mathematical problem

In this paper the simple electric circuit (a elementary RC charge circuit) shown in figure 1 below will be analyzed from the point of view of BV functions, in order to show how a simple structure like this give rise to a well defined and meaningful product of distributions. The element labelled V_S is a *constant voltage generator* (practically a high-stability battery) with output voltage V_S, S is a *ideal switch* (i.e. a switch with *negligible turn-on and turn-off times*), while R and C respectively are a *linear resistor* and a *linear capacitor*. Suppose now that the switch S closes for $t = 0$: then the voltage V_S is applied *istantaneously* to the branch made of the series connection of resistor R and capacitor C. A current $i(t) = i_C(t)$ starts to flow through resistor R and charges capacitor C: the voltage $v_C(t)$ across this circuit element rises until the difference $V_S - v_C(t)$ reaches 0. The description of the electrical regime in the network is then given by the *Kirchhoff voltage law*

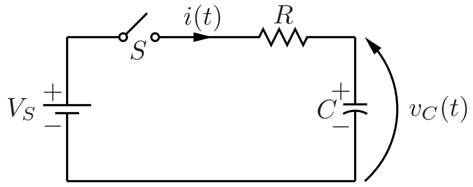

Figure 1: RC charge circuit schematic

$$V_S - Ri_C(t) - v_C(t) = V_S - RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} - v_C(t) = 0 \qquad t \geq 0$$

together with the knowledge of the voltage across (and therefore the stored charge) the capacitor C at the time $t < 0$

$$v_C(t)|_{t=0^-} = v_0$$

Assuming *zero charge on* C, the electrical regime is therefore described by the following non-homogeneous Cauchy problem

$$\begin{cases} RC\dfrac{\mathrm{d}v_C(t)}{\mathrm{d}t} - v_C(t) = V_s H(t) \\ v_C(t)|_{t=0^-} = 0 \end{cases} \tag{1}$$

where

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

is the ordinary *Heaviside function*. The solution of this problem has the following form

$$v_C(t) = \begin{cases} V_S\left(1 - e^{-\frac{t}{RC}}\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$

and consequently

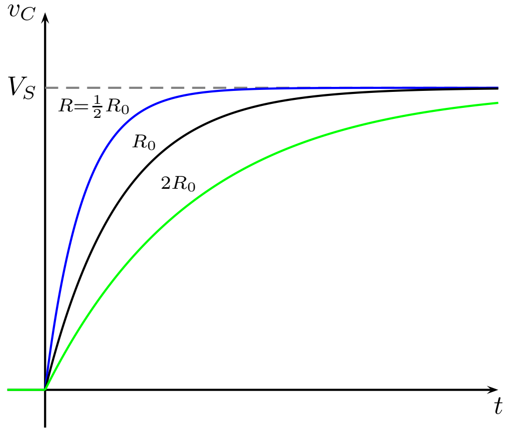

Figure 2: Plot of $v_C(t)$ for different values of R

$$i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = \begin{cases} \frac{V_S}{R}e^{-\frac{t}{RC}} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{3}$$

Known the time behavior of the voltage $v_C(t)$, it is sometimes necessary to know how much *energy* $\mathscr{E}_C$ is stored in the capacitor C at the end of the charging process, i.e. for $t = +\infty$. This means that it is necessary to evaluate the following integral

$$\mathscr{E}_C = \int_{-\infty}^{+\infty} v_C(t) i_C(t) \mathrm{d}t = C\int_{-\infty}^{+\infty} v_C(t) \mathrm{d}v_C(t) = C\int_{\mathbb{R}} v_C(t) \mathrm{d}v_C(t) \tag{4}$$

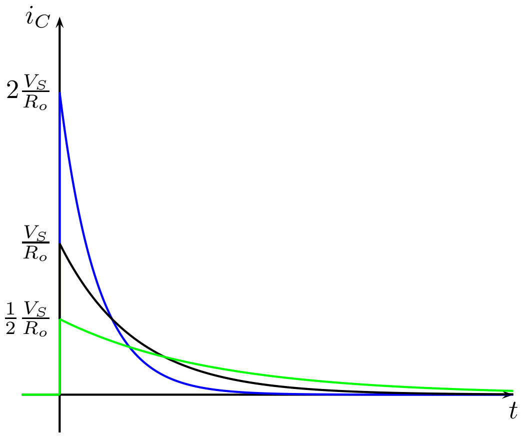

Figure 3: Plot of $i_C(t)$ for different values of R

It is exactly this quantity that, apart from having a precise meaning as a physical entity, has also a precise meaning in the space of BV functions as a *product of distributions*.

2 Analysis of the energy integral $\mathscr{E}_C$ in the space $BV(\mathbb{R})$

The works [3] and [4] are the principal references used in the following text: the Author has preferred their approach even if they deal extensively only with BV functions of *several variables*, since they contain a proof of the *composition formula* for BV functions which is needed to define the product of distribution implied by the structure of the energy integral $\mathscr{E}_C$.

First of all, a definition of what a BV function is is needed (see pages 158-161 of reference [4]): functions of bounded variations are functions whose *first derivative in the sense of distributions* (a classical reference is [5], page 34) is bounded.

Definition 1 *A real-variable function u is said to be* of bounded variation *and write $u \in BV(\mathbb{R})$ if and only if there is a* Radon measure $\mathrm{d}u \in \mathcal{M}(\mathbb{R})$ *such that*

$$\int_{\mathbb{R}} u(t)\dot{\varphi}(t)\mathrm{d}t = -\int_{\mathbb{R}} \varphi(t)\mathrm{d}u(t)$$

for all functions $\varphi \in C_0^1(\mathbb{R})$ (obviously $\dot{\varphi} \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}t}$).

Since

$$\int_{\mathbb{R}} v_C(t)\dot{\varphi}(t)\mathrm{d}t = -\frac{1}{C}\int_{\mathbb{R}} \varphi(t) i_C(t)\mathrm{d}t = -\frac{1}{C}\int_0^{+\infty} \varphi(t) i_C(t)\mathrm{d}t$$

the solution (2) of problem (1) is a *function of bounded variation for every value of the resistance R from zero to $+\infty$*. To see this, let $\mathrm{supp}\varphi(t) = [a,b] \in \mathbb{R}$ be the support of the function $\varphi(t)$: then it is easy to see that

$$\int_{\mathbb{R}} v_C(t)\dot{\varphi}(t)\mathrm{d}t = \begin{cases} 0 &]0,+\infty[\cap[a,b]=\varnothing \\ -\frac{1}{C}\int_a^b \varphi(t) i_C(t)\mathrm{d}t & a>0 \\ -\frac{1}{C}\int_0^b \varphi(t) i_C(t)\mathrm{d}t & a\leq 0 \end{cases}$$

When $R>0$, by using the *Mean-Value theorem* (see for example reference [2], pages 689) it is possible to estimate those integrals:

$$\begin{aligned} \int_{\mathbb{R}} v_C(t)\dot{\varphi}(t)\mathrm{d}t &= \varphi\left(\bar{t}\,\right)\int_a^b -\frac{V_S}{RC}e^{-\frac{t}{RC}}\mathrm{d}t = \\ &= \varphi\left(\bar{t}\,\right)\left[V_S e^{-\frac{t}{RC}}\right]_a^b = V_S\varphi\left(\bar{t}\,\right)\left(e^{-\frac{b}{RC}}-e^{-\frac{a}{RC}}\right) \end{aligned} \tag{5}$$

for $a>0$ with $\bar{t}\in[a,b]$ a proper fixed point, and

$$\begin{aligned} \int_{\mathbb{R}} v_C(t)\dot{\varphi}(t)\mathrm{d}t &= \varphi\left(\bar{t}\,\right)\int_0^b -\frac{V_S}{RC}e^{-\frac{t}{RC}}\mathrm{d}t = \\ &= \varphi\left(\bar{t}\,\right)\left[V_S e^{-\frac{t}{RC}}\right]_0^b = V_S\varphi\left(\bar{t}\,\right)\left(e^{-\frac{b}{RC}}-1\right) \end{aligned} \tag{6}$$

for $a\leq 0$ with $\bar{t}\in[0,b]$ a proper fixed point.

When $R=0$, remembering that $BV(\mathbb{R})$ is a Banach space (see references [4] pages 158-152), every Cauchy sequence in this space has a limit belonging to it, therefore it suffices to evaluate *the limit* of the above integral for R going to 0. The integral (5), where $0\notin[a,b]$, goes to 0 for R going to 0 since $V_S\varphi(\bar{t}\,)$ is bounded. Evaluation of the integral (6), where $0\in[a,b]$, needs a little more work: by using the Mean-Value theorem and choosing a proper *time value* $k\sqrt{RC}\in]\varepsilon,b-\varepsilon[$ for $\varepsilon>0$, this integral can be decomposed into two integrals on the intervals $[0,k\sqrt{RC}]$ and $[k\sqrt{RC},b]$. Therefore

$$\begin{aligned} \int_{\mathbb{R}} v_C(t)\dot{\varphi}(t)\mathrm{d}t &= \varphi\left(\bar{t}_1\right)\int_0^{k\sqrt{RC}} -\frac{V_S}{RC}e^{-\frac{t}{RC}}\mathrm{d}t + \varphi\left(\bar{t}_2\right)\int_{k\sqrt{RC}}^b -\frac{V_S}{RC}e^{-\frac{t}{RC}}\mathrm{d}t = \\ &= \varphi\left(\bar{t}_1\right)\left[V_S e^{-\frac{t}{RC}}\right]_0^{k\sqrt{RC}} + \varphi\left(\bar{t}_2\right)\left[V_S e^{-\frac{t}{RC}}\right]_{k\sqrt{RC}}^b = \\ &= V_S\left[\varphi\left(\bar{t}_1\right)\left(e^{-\frac{k}{\sqrt{RC}}}-1\right)+\varphi\left(\bar{t}_2\right)\left(e^{-\frac{b}{RC}}-e^{-\frac{k}{\sqrt{RC}}}\right)\right] \end{aligned}$$

where $\varphi\left(\bar{t}_1\right)\in[0,k\sqrt{RC}]$ and $\varphi\left(\bar{t}_2\right)\in[k\sqrt{RC},0]$: since

$$\lim_{R\to 0}\varphi\left(\bar{t}_1\right)e^{-\frac{k}{\sqrt{RC}}} = \lim_{R\to 0}\varphi\left(\bar{t}_2\right)e^{-\frac{k}{\sqrt{RC}}} = \lim_{R\to 0}\varphi\left(\bar{t}_2\right)e^{-\frac{b}{RC}} = 0$$

and since $\varphi\left(\bar{t}_1\right)$ tends to $\varphi(0)$ because $[0,k\sqrt{RC}]$ tends $\{0\}$ as R tends to 0, the limit of the given integral is

$$\int_{\mathbb{R}} v_C(t)\dot{\varphi}(t)\mathrm{d}t = -V_S\varphi(0)$$

therefore

$$\mathrm{d}v_C(t) = V_S\delta(t)\mathrm{d}t \tag{7}$$

i.e. *the limit of the weak derivative of the voltage $v_C(t)$ as R tends to zero is the Dirac measure* multiplied by a constant term V_S (see reference [5] page 36).

Now it is possible to deal with the basic problem stated in the introduction: first of all it is necessary to note that if $u(t)$ is a BV function of *one variable* the following limits exists and are *finite*

- $u_+(t)=\lim_{s\to t+} u(s)$, the *right limit* of the function u at the point $t \in \mathbb{R}$
- $u_-(t)=\lim_{s\to t-} u(s)$, the *left limit* of the function u at the point $t \in \mathbb{R}$

For a proof of this assertion, see references [3], pages 237-243, and [4], pages 161-164: this fact implies that BV functions of one variable have only *jump-type singularities* and also that the following definition is well-posed.

Definition 2 *Let $f : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function. Given a function $u \in BV(\mathbb{R})$ the* averaged superposition *is defined as*

$$\bar{f}\big(u(t)\big)= \int_0^1 f\big(u_+(t)s + u_-(t)(1-s)\big)\mathrm{d}s \tag{8}$$

Acknowledged the concept of averaged superposition, it is possible to formulate the following

Theorem 1 *Let $f : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable function. Given functions $\boldsymbol{u}(t) = (u_1(t), \dots, u_p(t)) \in \big(BV(\mathbb{R})\big)^p$ and let the averaged superposition $\partial \bar{f}\big(\boldsymbol{u}(t)\big)/\partial u_k$ be locally summable respect to the Radon measure $\mathrm{d}u_k(t)/\mathrm{d}t$ for $k = 1, \dots, p$. Then $f\big(\boldsymbol{u}(t)\big) \in BV(\mathbb{R})$ and the formula*

$$\frac{\mathrm{d}f\big(\boldsymbol{u}(t)\big)}{\mathrm{d}t} = \sum_{k=1}^{k=p} \frac{\partial \bar{f}\big(\boldsymbol{u}(t)\big)}{\partial u_k} \frac{\mathrm{d}u_k(t)}{\mathrm{d}t} \tag{9}$$

holds

For a proof of this important theorem in the multi-dimensional setting see reference [3], pages 248-250. Then, since $v_C(t) = V_S H(t)$ and $\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = V_S \delta(t)$ in the limiting case $R = 0$, considering the function $f\big(u(t)\big)= u^2(t)$ formula 9 reads as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C^2(t) = 2\bar{v}_C(t)\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = 2 \cdot V_S \bar{H}(t) \cdot V_S \delta(t) = 2V_S^2 \bar{H}(t) \cdot \delta(t) \tag{10}$$

having defined

$$\bar{H}(t) = \frac{1}{2}[H_+(t) + H_-(t)] = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

and since

$$\begin{aligned} \frac{\partial \bar{f}\big(v_C(t)\big)}{\partial u} = 2\bar{v}_C(t) &= 2\int_0^1 \big(v_{C+}(t)s + v_{C-}(t)(1-s)\big)\mathrm{d}s = \\ &= 2\left(v_{C+}(t) \left[\frac{1}{2}s^2\right]_0^1 - v_{C-}(t) \left[\frac{1}{2}(1-s)^2\right]_0^1 \right) = \\ &= [v_{C+}(t) + v_{C-}(t)] \end{aligned}$$

It is now possible to give the following definition:

$$V_S H(t)\cdot V_S\delta(t) \overset{def}{=} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}v_C^2(t) = \frac{V_S^2}{2}\cdot\delta(t) \Longleftrightarrow H(t)\cdot\delta(t) = \frac{1}{2}\delta(t) \tag{11}$$

this product of distribution is well-defined and has also a very simple physical mean: since for each $R \geq 0$ the following equality holds true

$$\mathscr{E}_C = C\int_{\mathbb{R}} v_C(t)\mathrm{d}v_C(t) = C\int_0^{+\infty} v_C(t)\mathrm{d}v_C(t) = \frac{1}{2}CV_S^2$$

then equation (11) represents the power sinked by (or the power sourced by the generator V_S to) the capacitor C in the limiting case of zero resistance of the charging circuit. Note that it is possible to give the same definition of this product even by using only the *Leibnitz formula* (proved in reference [4], pages 189-191) i.e. formula (9) for the function $f(u_1(t), u_2(t)) = v_C(t)u(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}v_C(t)u(t) = \bar{u}(t)\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \bar{v}_C(t)\frac{\mathrm{d}u(t)}{\mathrm{d}t}$$

and this is a consequence of the coherent structure of the calculus of BV functions.

As a last observation for this section, it is worth noting that the *total energy* $\mathscr{E}_S$ sourced by the generator V_S has the following expression

$$\mathscr{E}_S = \int_{-\infty}^{+\infty} V_S(t)i_C(t)\mathrm{d}t = C\int_{\mathbb{R}} V_S(t)\mathrm{d}v_C(t) = CV_S\int_0^{+\infty}\mathrm{d}v_C(t) = CV_S^2$$

whenever $R > 0$ or $R = 0$. This means that *apparently, half of the total energy sourced by the generator V_S has disappeared during the charging process in the limiting case* $R = 0$: when $R > 0$ the "lost" energy is dissipated in the resistor R as Joule's Law predicts.

3 Conclusions

The consequences of the analysis done in the preceding sections are essentially two:

1. A formula for the distributional product of the Heaviside function and the Dirac distribution was given and it was shown that this formula has a very simple physical meaning. It is worth to notice that the formula was obtained by means of "classical analysis" in the function space $BV(\mathbb{R})$, without recourse to the methods of the theory of distributions.

2. It was shown that the given formula implies that *the model physical system violates the energy balance principle* for the model physical system (a simple RC charge circuit). This is only due to the fact that for the limiting case $R = 0$ this model is no more valid: it is necessary to use the full *Maxwell system* to determine the electrical regime of this system, and therefore to know its *geometrical shape.* Precisely, the *energy balance principle* for a electromagnetic system looks like

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{tot}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{cond}} + \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{rad}} \tag{12}$$

   where

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{tot}}$ is the *total power* generated and dissipated by the system,

(b) $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{cond}}$ is the power generated and/or dissipated by physical processes involving conduction or more generally *movements of charges in solids*,

(c) $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{rad}}$ is the *radiated power* i.e. the *flux of the Poynting vector* $\boldsymbol{S}$ (see for example [1]) flowing across a surface Σ of arbitrary shape and given *normal vector* $\hat{\boldsymbol{n}}$, enclosing the given physical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{rad}} = \int_{\Sigma} \boldsymbol{S} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}\sigma$$

This second term of equation (12) is *barely neglected* when using a *circuital i.e. quasi-stationary description* of the system, and this is no more possible in the limiting case $R = 0$.

The author asks if the second occurrence described above is to be found in every model system where a product of distributions arises which violates a *energy balance principle*, i.e if there is a more complex physical description giving rise to an *integral balance law* where this product appears together with a neglected quantity, restoring equality.

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