Mathematical analysis of static characteristics of the 1-D MOS capacitor

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Abstract

In this work, a complete analytic description of the electric field and its electrostatic potential in the one-dimensional MOS (Metal Oxide Semiconductor) capacitor is given: no claim is made about the originality of contents, nor about about the completeness of references. However, the Author has decided to collect and eventually reconstruct known results since a comprehensive reference on the subject lacks.

1 The mathematical model of the 1-D MOS capacitor

1.1 The general mathematical model of electrostatic problems

The device model consist of a *3-dimensional physical region* $\boldsymbol{\Omega}$ whose boundary is the *surface* $\flat\boldsymbol{\Omega}$, having an almost everywhere defined *set-theoretic inward normal vector* $\vec{\boldsymbol{n}}$ (see for example [7], page 227): our goal is to determine the *stationary electric field* $\boldsymbol{E}$ and its *electrostatic potential* ϕ inside that region, assuming that the *magnetic induction* $\boldsymbol{B}$ is zero. *Faraday's law* (see for example [3], pages 9-11, but any good textbook on electromagnetic theory suffices) implies that

$$\nabla\times\boldsymbol{E}(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{E}(\boldsymbol{x}) = -\nabla\phi(\boldsymbol{x}) \quad \boldsymbol{x}\in\boldsymbol{\Omega} \tag{1}$$

(the minus sign is conventional) while *Gauss' law in matter* reads as follows

$$\nabla\cdot\boldsymbol{D}(\boldsymbol{x}) = \varrho(\phi,\boldsymbol{x}) \qquad \boldsymbol{x}\in\boldsymbol{\Omega} \tag{2}$$

where $\boldsymbol{D}$ is the *electric displacement*, expressed by its *constitutive equation*

$$\boldsymbol{D}(\boldsymbol{x}) = \boldsymbol{\mathfrak{D}}(\boldsymbol{E},\boldsymbol{x}) \qquad \boldsymbol{x}\in\boldsymbol{\Omega} \tag{3}$$

not in its most general form, since we assume its independence on the magnetic induction $\boldsymbol{B}$, and ϱ is the *electric charge density* inside the device, which is required to satisfy the following *neutrality condition*:

$$\varrho(0,\boldsymbol{x}) = 0 \qquad \boldsymbol{x}\in\boldsymbol{\Omega} \tag{4}$$

Combining equations (1), (2) and (3) the following equation for the electrostatic potential is obtained

$$\nabla\cdot \boldsymbol{\mathfrak{D}}(-\nabla\phi, \boldsymbol{x}) = \varrho(\phi, \boldsymbol{x}) \qquad \boldsymbol{x} \in \boldsymbol{\Omega} \tag{5}$$

Note that this equation has to be intended in the *weak sense* (see for example reference [6], page 67), since the second member of the constitutive equation (3) may have various kind of discontinuities, due for example to interfaces between different materials and other physical phenomena (see reference [3], pages 27-33, for a comprehensive description). This equation has to be solved in conjunction with the following specialized *Robin boundary condition*

$$\phi(\boldsymbol{s}) \cdot \sum_{i=1}^{n} \chi_{\Gamma_i}(\boldsymbol{s}) + \frac{\partial \phi}{\partial \vec{\boldsymbol{n}}}(\boldsymbol{s}) \cdot \chi_{\flat\boldsymbol{\Omega}\setminus\cup_{i=1}^{n}\Gamma_i}(\boldsymbol{s}) = \sum_{i=1}^{n} V_i \chi_{\Gamma_i}(\boldsymbol{s}) \quad \boldsymbol{s} \in \flat\boldsymbol{\Omega} \tag{6}$$

where

- the quantity

  $$\frac{\partial \phi}{\partial \vec{\boldsymbol{n}}}(\boldsymbol{s}) = \langle \vec{\boldsymbol{n}}(\boldsymbol{s}), \nabla\phi(\boldsymbol{s})\rangle = -\langle \vec{\boldsymbol{n}}(\boldsymbol{s}), \boldsymbol{E}(\boldsymbol{s})\rangle = -\boldsymbol{E}_{\vec{\boldsymbol{n}}}(\boldsymbol{s}) \quad \boldsymbol{s} \in \flat\boldsymbol{\Omega} \setminus \cup_{i=1}^{n}\Gamma_i$$

  is the *electric field normal to the insulating part of the boundary*

- $\chi_{\{\cdot\}}$ are *characteristic functions* of the sets $\{\cdot\}$, i.e.

  $$\chi_{\{\cdot\}}(\boldsymbol{s}) = \begin{cases} 1 & \text{if } \boldsymbol{s} \in \{\cdot\} \\ 0 & \text{if } \boldsymbol{s} \notin \{\cdot\} \end{cases}$$

- the *pairwise disjoint* compact boundary regions $\Gamma_i \Subset \flat\boldsymbol{\Omega}$, $i = 1, \ldots, n \geq 2$ are *perfectly conducting contacts* (see [3], page 14), where the *constant voltages* V_i, $i = 1, \ldots, n \geq 2$ are assigned. *The compactness requirement is a physical one that can be dropped on simplified device models*, like the 1-D MOS capacitor considered here, as we will see later.

Simply stated, condition (6) requires that (Dirichlet type condition)

$$\phi = V_i \text{ on each contact } \Gamma_i,\ i = 1, \ldots, n \geq 2$$

and (Neumann type condition)

$$\boldsymbol{E}_{\vec{\boldsymbol{n}}} = 0 \text{ on the } \textit{insulating part} \text{ of the surface } \flat\boldsymbol{\Omega} \setminus \cup_{i=1}^{n}\Gamma_i$$

meaning that *at least one potential difference is applied to the device by a couple of contacts* and *charges cannot cross nor store on the insulating part of device's boundary.* The requirement $n \geq 2$ in the Dirichlet condition *is essential* since *it implies that there is at least one non-trivial solution to equation* (5): the author has not been able to find a precise source for the mathematical proof of this physical fact.

1.2 Basic assumptions and construction of the 1-D MOS capacitor model

Now the preceeding general description has to be adapted to describe the 1-D MOS capacitor: let's assume the following hypothesis.

1. Assume that $\boldsymbol{\Omega}$ is the space between two perfectly conducting parallel planes Γ_1 and Γ_2 i.e. an unbounded 3-dimensional region of constant thickness d, allowing also $d = +\infty$. This hypothesis, jointly with the above requirement $i \geq 2$, implies
$$\Gamma_1 \cup \Gamma_2 = \flat\boldsymbol{\Omega}$$
This also implies that *the usual hypothesis of compactness of the contacts should be dropped.*

2. Choose a cartesian frame of reference $(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) \in (\mathbb{R}^3)^3$ in such a way that
$$\boldsymbol{x} = \big(\langle \boldsymbol{x}, \vec{\boldsymbol{e}}_1\rangle, \langle \boldsymbol{x}, \vec{\boldsymbol{e}}_2\rangle, \langle \boldsymbol{x}, \vec{\boldsymbol{e}}_3\rangle\big) = (x, y, z)$$
and that the preceeding two planes can be expressed as follows:
$$\Gamma_1 = \{\boldsymbol{s} = (0, y, z) \in \boldsymbol{\Omega} \,|\, y, z \in \mathbb{R}\}$$
and
$$\Gamma_2 = \{\boldsymbol{s} = (d, y, z) \in \boldsymbol{\Omega} \,|\, y, z \in \mathbb{R}\}$$

3. Assume that there is a positive number $t_{ox} \ll d$ (*oxide thickness*) and a third plane Γ_{ox}
$$\Gamma_{ox} = \{\boldsymbol{x} = (t_{ox}, y, z) \in \boldsymbol{\Omega} \,|\, y, z \in \mathbb{R}\}$$
such that in the device volume $\boldsymbol{\Omega}$ one can identify two different regions, respectively the *oxide region*
$$\boldsymbol{\Omega}_{ox} = \{\boldsymbol{x} \in \boldsymbol{\Omega} \,|\, 0 < x = \langle \boldsymbol{x}, \vec{\boldsymbol{e}}_1\rangle < t_{ox}\}$$
and the *semiconductor region*
$$\boldsymbol{\Omega}_{s} = \{\boldsymbol{x} \in \boldsymbol{\Omega} \,|\, t_{ox} < x = \langle \boldsymbol{x}, \vec{\boldsymbol{e}}_1\rangle < d\}$$
where the electric displacement is *linear* respect to the electric field or, which is the same in our stationary case, respect to the gradient of the electric potential
$$\boldsymbol{D}(\boldsymbol{x}) = \mathfrak{D}(-\nabla\phi, \boldsymbol{x}) = \begin{cases} -\varepsilon_{ox}\nabla\phi(\boldsymbol{x}) & \boldsymbol{x} \in \boldsymbol{\Omega}_{ox} \\ -\varepsilon_{s}\nabla\phi(\boldsymbol{x}) & \boldsymbol{x} \in \boldsymbol{\Omega}_{s} \end{cases} \tag{7}$$
where the *strictly positive* constants ε_{ox} and ε_s are respectively the *oxide permittivity* and *semiconductor permittivity*

4. Assume that the charge density ρ has the following form (see for example [2], page 849):
$$\varrho(\phi, \boldsymbol{x}) = \begin{cases} 0 & \boldsymbol{x} \in \boldsymbol{\Omega}_{ox} \\ q\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi(\boldsymbol{x})}{V_T}} - n_{p0}e^{\frac{\phi(\boldsymbol{x})}{V_T}}\right] & \boldsymbol{x} \in \boldsymbol{\Omega}_{s} \end{cases} \tag{8}$$
where

- $q \simeq 1.602 \cdot 10^{-19}\,\mathrm{C}$ is the *elementary charge*
- $N_A^{(0)}$ is the *acceptor doping density* (see for example [2], pages 633-651) i.e. the *volume density* (m^{-3}) of acceptor atoms in the semiconductor: specification of this single parameter implies that $\boldsymbol{\Omega}_s$ is a *p-type semiconductor*. Note also that the *neutrality condition* (4) implies
$$N_A^{(0)} = n_{p0} - p_{p0} \leq 0 \tag{9}$$
- p_{p0} is the *hole equilibrium density*
- n_{p0} is the *electron equilibrium density*
- $V_T = \frac{k_B T}{q}$ is the *electron thermal voltage*: T (K) is the *absolute temperature* and $k_B \simeq 1.380 \cdot 10^{-23}\,\mathrm{J/K}$ is the *Boltzmann constant*

As a consequence of the four previous hypothesis we have

$$\frac{\partial \phi(\boldsymbol{x})}{\partial y} = \frac{\partial \phi(\boldsymbol{x})}{\partial z} = 0$$

and

$$\phi(\boldsymbol{x}) = \phi(\langle \boldsymbol{x}, \vec{\boldsymbol{e}}_1 \rangle) = \phi(x)$$

Thus the general problem of the solution of equation (5) with condition (6) reduces to the following boundary value problem for a non linear ordinary differential equation with discontinuous step-like coefficients

$$\begin{cases} -\varepsilon_{ox} \frac{\mathrm{d}^2 \phi(x)}{\mathrm{d}x^2} = 0 & 0 < x \leq t_{ox} \\ -\varepsilon_s \frac{\mathrm{d}^2 \phi(x)}{\mathrm{d}x^2} = q \left[N_A^{(0)} + p_{p0} e^{-\frac{\phi(x)}{V_T}} - n_{p0} e^{\frac{\phi(x)}{V_T}} \right] & t_{ox} \leq x < d \end{cases} \tag{10}$$

$$\begin{cases} \phi(0) = V_G & \text{Dirichlet condition at the gate} \\ \phi(d) = 0 & \text{Dirichlet condition at the bulk} \\ \langle \vec{\boldsymbol{e}}_1, \boldsymbol{D}_{ox}^- \rangle = \langle \vec{\boldsymbol{e}}_1, \boldsymbol{D}_s^+ \rangle & \text{junction condition at } x = t_{ox} \end{cases} \tag{11}$$

where the *junction condition data* relates the electric displacement fields in oxide and semiconductor and is a mean to use equation (5) in a classical (i.e. not weak) formulation. The junction condition permit to understand the behavior of the potential across surfaces (or points) of discontinuity: it has the following meaning

$$\langle \vec{\boldsymbol{e}}_1, \boldsymbol{D}_{ox}^- \rangle = \lim_{x \to t_{ox}^-} \varepsilon_{ox} \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \qquad \langle \vec{\boldsymbol{e}}_1, \boldsymbol{D}_s^+ \rangle = \lim_{x \to t_{ox}^+} \varepsilon_s \frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$$

and is comprehensively examined, in reference [3] pages 32-34. In what follows, problem (10), (11) will be called the *Dirichlet problem for the Poisson equation of the MOS capacitor.*

2 General solution of the Poisson equation for the 1-D MOS capacitor in implicit form

2.1 Qualitative properties of the solution

Since problem (10), (11) is a *boundary value problem* and moreover has *discontinuous* (precisely *step-like*) coefficients, the classical theorems on existence and uniqueness of solution are not applicable. However, the potential $\phi(x)$ enjoys some qualitative properties as a consequence of the form of its defining equation (10) and of the kind of boundary and junction conditions (11) it satisfies. Precisely,

1. *$\phi(x)$ is differentiable of class* $C^2([0, t_{ox}[\cup]t_{ox}, d])$, i.e. it is two times differentiable with continuous second derivative for each $x \in [0, t_{ox}]\cup]t_{ox}, d]$,

   This is a trivial consequence of the classical definition of solution to problem (10). The potential $\phi(x)$ at the point $x = t_{ox}$ fails even to be of class C^1 whenever $\varepsilon_{ox} \neq \varepsilon_s$, due to the *junction condition* imposed there by the laws of electromagnetism: it could only be, and in fact is as we will see later, *continuous* there.

2. *$\phi(x)$ attains its maximum and minimum values on the boundary points $x = 0$ and $x = d$ and it is strictly monotone.*

   To prove this, first apply the *one-dimensional maximum principle* (theorem 1, pages 2-3 of reference [5]) to region $]0, t_{ox}[$, where equation (10) is *linear*, and see that $\phi(x)$ *cannot have a local maximum or minimum there.* This has two consequences:

   - the *continuous first derivative* $\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$ is never zero inside that region nor changes its sign, so $\phi(x)$ is *strictly monotone* there.
   - the points $x = 0$ and $x = t_{ox}$ are the only points where $\phi(x)$ can attain a local maximum or minimum.

   Second, apply the *one-dimensional maximum principle for non linear equations* (theorem 21, page 48 of reference [5]) to equation (10) in region $]t_{ox}, d[$. Noting that $\phi(x) \equiv 0$ is a particular solution of this equation, the following results, completely analogous to previous ones, are easily proved

   - the *continuous first derivative* $\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$ is never zero inside that region nor changes its sign, so $\phi(x)$ is *strictly monotone* there.
   - the points $x = t_{ox}$ and $x = d$ are the only points where $\phi(x)$ can have a local maximum or minimum.

   Third, note that, by the *junction condition*, $\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$ *does not change sign in a neighborhood of $x = t_{ox}$*: then, by entirely elementary facts about differentiable strictly monotone real functions of a real variable (see for example theorem 6.1, pages 322-323 of reference [4]) its easily to conclude that

   - if $V_G = \phi(0) > 0$ then $x = 0$ and $x = d$ are respectively the *absolute maximum and minimum of $\phi(x)$*: the two points exchange their role if $V_G < 0$. The estimate

     $$|\phi(x)| \leq |V_G| \qquad \forall x \in [0, d] \tag{12}$$

     holds universally
   - *$\phi(x)$ is strictly monotone decreasing if $V_G > 0$ strictly monotone increasing if $V_G < 0$*, on its *whole domain of definition* $[0, d]$.

3. If $\phi(x)$ exists, *it is unique*. This result for $0 < d < +\infty$ follows from two theorems of reference [5]: precisely, for the region $[0, t_{ox}]$ this is a consequence of theorem 8, page 13, while for the region $[t_{ox}, d]$ this is a consequence of theorem 22, page 48. *Note that the cases analyzed in the given reference are exactly ours*, being possible for the condition at one extremity of the domain to be the *boundary value of the first derivative of the unknown function*: the case $d = +\infty$ can be treated as a limiting case for d *finite but increasing towards* $+\infty$.

4. $\phi(x)$ *is continuous on* $x = t_{ox}$: jointly with the first regularity result of this section, this implies that *it is of class* $C^2([0, t_{ox}[\cup]t_{ox}, d]) \cap C^0[0, d]$.

   To prove this assertion, first note that if we define ϕ_s^+ and ϕ_s^- as the following quantities,

$$\phi_s^+ = \lim_{\epsilon \to 0^+} \phi(t_{ox} + \epsilon) \qquad \phi_s^- = \lim_{\epsilon \to 0^-} \phi(t_{ox} - \epsilon)$$

   called respectively *right limit* and *left limit* of the potential as x approaches t_{ox}, then one of the following relations holds true for all $\epsilon > 0$

$$\begin{aligned} \phi(t_{ox} - \epsilon) \geq \phi_s^- \geq \phi_s^+ \geq \phi(t_{ox} + \epsilon) \qquad & V_G > 0 \\ \phi(t_{ox} - \epsilon) \leq \phi_s^- \leq \phi_s^+ \leq \phi(t_{ox} + \epsilon) \qquad & V_G < 0 \end{aligned}$$

   since $\phi(x)$ is strictly monotone and therefore *the only kind of singularity the potential can have at* $x = t_{ox}$ *is a jump* (see for example reference [8], page 169). This implies

$$0 \leq \left| \phi_s^+ - \phi_s^- \right| < +\infty$$

   Next step of proof is to express equation (10) in its *weak form* (see for example reference [6], page 67) locally near the point $x = t_{ox}$, and estimate its members using the usual L^1-norm. If $\Delta_\epsilon = [t_{ox} - \epsilon, t_{ox} + \epsilon]$ is the ϵ-neighborhood of that point, the needed equation is precisely the following

$$\begin{aligned} &\int_{\Delta_\epsilon} \varepsilon(x) \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} \mathrm{d}x \\ &\qquad = q \int_{\Delta_\epsilon} \left[N_A^{(0)} + p_{p0} e^{-\frac{\phi(x)}{V_T}} - n_{p0} e^{\frac{\phi(x)}{V_T}} \right] \varphi(x) \mathrm{d}x \end{aligned} \tag{13}$$

   where $\varphi(x)$ is an *arbitrary function* of class $C_0^1([t_{ox} - \epsilon, t_{ox} + \epsilon])$ i.e. a differentiable function vanishing on the endpoints and outside its interval of definition, and

$$\varepsilon(x) = \begin{cases} \varepsilon_{ox} & x \in [0, t_{ox}] \\ \varepsilon_s & x \in]t_{ox}, d] \end{cases}$$

   In order to compare easily the two members of equation (13), we would like them to share the same structure: this can be accomplished by transforming the second member as follows

$$\begin{aligned} &q \int_{\Delta_\epsilon} \left[N_A^{(0)} + p_{p0} e^{-\frac{\phi(x)}{V_T}} - n_{p0} e^{\frac{\phi(x)}{V_T}} \right] \varphi(x) \mathrm{d}x \\ &\qquad = q \int_{\Delta_\epsilon} \left\{ \int_{t_{ox} - \epsilon}^{x} \left[N_A^{(0)} + p_{p0} e^{-\frac{\phi(y)}{V_T}} - n_{p0} e^{\frac{\phi(y)}{V_T}} \right] \mathrm{d}y \right\} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} \mathrm{d}x \end{aligned}$$

Now the second member share the same structure of the first: then, since $\varepsilon(x)\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$ is integrable, applying first the L^1-norm and then the *Hölder's Inequality* to both members (see for example reference [8], page 100), we can simplify the term depending only on the test function and obtain the following fundamental equality:

$$\left\|\varepsilon(x)\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right\|_{L^1(\Delta_\epsilon)} = \left\|q\int_{t_{ox}-\epsilon}^{x}\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi(y)}{V_T}} - n_{p0}e^{\frac{\phi(y)}{V_T}}\right]\mathrm{d}y\right\|_{L^1(\Delta_\epsilon)}$$

Next step is to estimate the two members of the equality as ϵ goes to 0: since $\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$ *does not change sign on* $[0, d]$ then the following inequality holds true for the first member

$$\begin{aligned}\lim_{\epsilon\to 0}\left\|\varepsilon(x)\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right\|_{L^1(\Delta_\epsilon)} &\geq \lim_{\epsilon\to 0}\varepsilon_{\mathrm{min}}\mathrm{sign}\left(-V_G\right)\left[\phi(x)\right]_{t_{ox}-\epsilon}^{t_{ox}+\epsilon} \\ &= \varepsilon_{\mathrm{min}}\mathrm{sign}\left(-V_G\right)\left[\phi_s^+ - \phi_s^-\right] \\ &= \varepsilon_{\mathrm{min}}\left|\phi_s^+ - \phi_s^-\right| \geq 0\end{aligned} \tag{14}$$

where

$$\varepsilon_{min} = \min\{\varepsilon_{ox}, \varepsilon_s\} > 0$$

is the *minimum permittivity*,

$$\mathrm{sign}\left(A\right) = \begin{cases} +1 & \text{if } A > 0 \\ 0 & \text{if } A = 0 \\ -1 & \text{if } A < 0 \end{cases}$$

is the *sign function*. The estimation of the second member is somewhat easier:

$$\begin{aligned}&\left\|q\int_{t_{ox}-\epsilon}^{x}\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi(y)}{V_T}} - n_{p0}e^{\frac{\phi(y)}{V_T}}\right]\mathrm{d}y\right\|_{L^1(\Delta_\epsilon)} \leq \\ &\leq \left\|q\int_{t_{ox}-\epsilon}^{x}\left|N_A^{(0)} + p_{p0}e^{-\frac{\phi(y)}{V_T}} - n_{p0}e^{\frac{\phi(y)}{V_T}}\right|\mathrm{d}y\right\|_{L^1(\Delta_\epsilon)} \leq 4\epsilon^2\cdot k(V_G)\end{aligned} \tag{15}$$

where $k(V_G)$ is a positive constant depending only on V_G, generally not sharp. Now joining together inequality (14) and inequality (15) we obtain the following equation

$$0 = \lim_{\epsilon\to 0} 4\epsilon^2\cdot k(V_G) \geq \varepsilon_{\mathrm{min}}\left|\phi_s^+ - \phi_s^-\right| \geq 0 \Leftrightarrow \phi_s^+ - \phi_s^- = 0 \tag{16}$$

The preceeding equation implies that

$$\phi_s^+ = \phi_s^- = \phi_s \tag{17}$$

which obviuously implies that $\phi(x)$ *continuous in* $x = t_{ox}$: actually that equation tells us more, but since we don't need it here we will not explore it further. The value ϕ_s plays an important role in what follows: it is the *oxide-semiconductor interface electrostatic potential.*

2.2 Analytic calculation of the solution in implicit form

In the region $0 \leq x \leq t_{ox}$ the solution of problem (10), (11) is *elementary*: assuming that the oxide-semiconductor interface electrostatic potential is known (while has to be determined), then

$$\phi(x) = V_G - \frac{V_G - \phi_S}{t_{ox}} x \qquad 0 \leq x \leq t_{ox} \tag{18}$$

The harder task is to solve the problem in the region $t_{ox} \leq x \leq d$: there the equation reads as follows

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = -\frac{q}{\varepsilon_s}\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi(x)}{V_T}} - n_{p0}e^{\frac{\phi(x)}{V_T}}\right] \quad t_{ox} \leq x < d \tag{19}$$

Now let us multiply both sides of equation (19) by $2\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$: then the first member become

$$2\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right)^2$$

while the second member become

$$-2\frac{q}{\varepsilon_s}\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi(x)}{V_T}} - n_{p0}e^{\frac{\phi(x)}{V_T}}\right]$$

Integrating the two new members on the interval $[t_{ox}, x]$ (with $x < d \leq +\infty$), we obtain the following results

$$\begin{aligned}
\int_{t_{ox}}^{x}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\right)^2\mathrm{d}y &= \left[\left(\frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\right)^2\right]_{y=t_{ox}}^{y=x} \\
&= \left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right)^2 - \left.\left(\frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\right)^2\right|_{y=t_{ox}} \\
&= \left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right)^2 - \frac{\varepsilon_{ox}}{\varepsilon_s}\left(\frac{V_G - \phi_s}{t_{ox}}\right)^2
\end{aligned}$$

while the second member

$$\begin{aligned}
-2\frac{q}{\varepsilon_s}\int_{t_{ox}}^{x}\frac{\mathrm{d}\phi(y)}{\mathrm{d}y}&\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi(y)}{V_T}} - n_{p0}e^{\frac{\phi(y)}{V_T}}\right]\mathrm{d}y = \\
&= -2\frac{q}{\varepsilon_s}\int_{\phi_s}^{\phi(x)}\left[N_A^{(0)} + p_{p0}e^{-\frac{\phi}{V_T}} - n_{p0}e^{\frac{\phi}{V_T}}\right]\mathrm{d}\phi \\
&= -2q\frac{V_T}{\varepsilon_s}\left[N_A^{(0)}\frac{\phi}{V_T} - p_{p0}e^{-\frac{\phi}{V_T}} - n_{p0}e^{\frac{\phi}{V_T}}\right]_{\phi_s}^{\phi(x)} \\
&= -2q\frac{V_T}{\varepsilon_s}\left[N_A^{(0)}\frac{\phi(x)}{V_T} - p_{p0}e^{-\frac{\phi(x)}{V_T}} - n_{p0}e^{\frac{\phi(x)}{V_T}}\right] - \mathscr{E}_S
\end{aligned}$$

where

$$\mathscr{E}_S = \mathscr{E}_S\left(N_A^{(0)}, \varepsilon_s, \phi_s\right) = -2q\frac{V_T}{\varepsilon_s}\left[N_A^{(0)}\frac{\phi_s}{V_T} - p_{p0}e^{-\frac{\phi_s}{V_T}} - n_{p0}e^{\frac{\phi_s}{V_T}}\right] \tag{20}$$

Now putting

$$\mathscr{E}_{MOS} = \mathscr{E}_{MOS}\begin{pmatrix} t_{ox} & \varepsilon_{ox} & V_G \\ N_A^{(0)} & \varepsilon_s & \phi_s \end{pmatrix} = \mathscr{E}_S - \frac{\varepsilon_{ox}}{\varepsilon_s}\left(\frac{V_G - \phi_s}{t_{ox}}\right)^2 \tag{21}$$

and putting together the two transformed members we obtain the following *first order algebraic differential equation*

$$\left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right)^2 = -2q\frac{V_T}{\varepsilon_s}\left[N_A^{(0)}\frac{\phi(x)}{V_T} - p_{p0}e^{-\frac{\phi(x)}{V_T}} - n_{p0}e^{\frac{\phi(x)}{V_T}}\right] - \mathscr{E}_{MOS} \tag{22}$$

Taking the square root of the two members and applying *Barrow's formula* (see for example [1], pages 18-20) we get the potential $\phi(x)$ in *implicit form*:

$$\int_{\phi_S}^{\phi(x)} \frac{\mathrm{d}\phi}{\pm\sqrt{-2q\frac{V_T}{\varepsilon_s}\left[N_A^{(0)}\frac{\phi}{V_T} - p_{p0}e^{-\frac{\phi}{V_T}} - n_{p0}e^{\frac{\phi}{V_T}}\right] - \mathscr{E}_{MOS}}} = x - t_{ox} \tag{23}$$

The plus sign $(+)$ in the denominator has to be used when $V_G < 0$ since in this occurrence the electric field is positive, while the minus sign $(-)$ has to be used when $V_G \geq 0$

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